On the Beta Function for Anisotropic Current Interactions in 2D

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By making use of current-algebra Ward identities we study renormalization of general anisotropic current-current interactions in 2D. We obtain a set of algebraic conditions that ensure the renormalizability of the theory to all orders. In a certain minimal prescription we compute the βeta function to all orders.

Left-right current-current interactions in 2D arise in many physical systems, for example in the Kosterlitz-Thouless transitions and in the study of free electrons in random potentials. The chiral Gross-Neveu model is the simplest example which is symmetry preserving (isotropic) [1]. Because such interactions are marginal in 2D, the couplings are dimensionless and there is usually no small parameter to expand in order to explore strong coupling.

In this work we determine the βeta function to all orders in a certain minimal prescription. This should be sufficient for the study of fixed points. The primary tool that allows us to isolate the log divergences at arbitrary order are the Ward identities for the currents. Kutasov performed a similar computation in the simpler isotropic case (i.e. non-abelian Thirring model, which is equivalent to the chiral Gross-Neveu model) but argued his result was the leading order in $1/k$ where k is the level of the current algebra [4]. We believe our result to be exact.

We do not expect that there is anything dramatically new to learn for the isotropic case. The behavior of models like the Gross-Neveu model is very well understood. This is in contrast to the anisotropic, i.e. symmetry breaking perturbations where one can expect richer phenomena. In the anisotropic case not all perturbations are in fact renormalizable. We find three renormalizability conditions that ensure the theory is renormalizable to all orders. Given these conditions are satisfied, we find a compact expression for the summation of all orders in perturbation theory.

We know of no reason to expect additional non-perturbative corrections to the βeta function due for instance to instantons. Our result should be a useful tool for exploring strong coupling physics for the many models in this class. In this Letter we only report the main result and defer applications to future publications.

Consider a conformal field theory with Lie algebra symmetry realized in the standard way [2,3]. It possesses left and right conserved currents, $J^a(z)$, $\bar{J}^a(\bar{z})$, where $z = x + iy$, $\bar{z} = x - iy$, satisfying the operator product expansion (OPE)

$$J^a(z)J^b(0) \sim \frac{k\eta^{ab}}{z^2} + \frac{1}{z}f_c^{ab}J^c(0) + \dots , \qquad (1)$$

and similarly for $\bar{J}^a(\bar{z})$. η^{ab} in the above equation is a metric (Killing form) and k is the level.

We include the case of Lie superalgebras with applications to disordered systems in mind. Here each current J^a has a grade $[a] = 0$ or 1, and the tensors satisfy: $\eta^{ab} = (-)^{[a][b]}\eta^{ba}$, $f^{abc} = -(-)^{[b][c]}f^{acb}$, $f_c^{ab} = -(-)^{[a][b]}f_c^{ba}$, where $f^{abc} = f_i^{ab}\eta^{ic}$. For superalgebras, η^{ab} is generally not diagonalizable, but we have $\eta_{ab}\eta^{bc} = \delta_a^c$.

The conformal field theory can be perturbed by marginal operators which are bilinear in the currents. The most general action is

$$S = S_{\text{cft}} + \int \frac{d^2x}{2\pi} \sum_A g_A \mathcal{O}^A , \quad \mathcal{O}^A = \sum_{a,\bar{a}} d_{a\bar{a}}^A J^a \bar{J}^{\bar{a}} , \quad (2)$$

where S_{cft} is the conformal field theory with the current-algebra symmetry, and $d_{a\bar{a}}^A$ are certain tensors that define the model. The simplest case is that of a single coupling where $d_{a\bar{a}} = \eta_{a\bar{a}}$ so that the perturbation is built on the Casimir and preserves the symmetry.

For these models the βeta function beyond 2 loops is prescription dependent. Let $\beta_g = b_2 g^2 + b_3 g^3 + \dots$. The prescription dependence is equivalent to a redefinition of the coupling $g' = g'(g)$. Let $g' = g + cg^2 + \dots$. One easily sees that $\beta'(g') = b_2 g'^2 + b_3 g'^3 + \dots$, so that the one- and two-loop coefficients b_2 and b_3 are universal. Though the higher loop contributions are prescription dependent, the existence of fixed points is not: if $\beta_g = 0$, then $\beta'(g') = \frac{\partial g'}{\partial g}\beta_g = 0$. For this reason, the particular prescription we will adopt is meaningful.

We will compute the βeta function in the following way. Let X be any arbitrary product of fields and suppose we isolate the ultra-violet logarithmic divergences in the following way:

$$\langle X \rangle = \sum_A F_A(g, \log a) \int \frac{d^2x}{2\pi} \left\langle \mathcal{O}^A(x)X \right\rangle + \dots . \quad (3)$$

where a is a short distance cutoff (lattice spacing). In general F_A has an expansion in powers of $\log a$:

$$F_A = -a_A^{(0)}(g) + \sum_{n=1}^{\infty} a_A^{(n)}(g) \log^n a . \qquad (4)$$

Requiring $dF_A/d\log a = 0$ gives

$$\beta_{g_A} \frac{dg_A}{d\log a} = G_{AB}^{-1} a_B^{(1)} , \quad G_{AB} = \partial_{g_B} a_A^{(0)} . \quad (5)$$

In our models equation (3) holds due to the Ward identities for the currents. This makes the computation much simpler that the two-loop computations in perturbed conformal field theory carried out in [5]. The order g^n term in the expansion is

$$\frac{(-)^n}{n!} g_{A_1} \ldots g_{A_n} d^{A_1}_{a_1 \bar{a}_1} \ldots d^{A_n}_{a_n \bar{a}_n}$$
$$\times \int d1 \ldots dn \left\langle J^{a_1} \bar{J}^{\bar{a}_1}(1) \ldots J^{a_n} \bar{J}^{\bar{a}_n}(n) X \right\rangle , \quad (6)$$

where $\int d1 = \int d^2 x_1 / 2\pi$. The Ward identities read

$$\left\langle J^{a_1}(1) \ldots J^{a_n}(n) X \right\rangle$$
$$= \sum_{i \neq 1} \frac{k \eta^{a_1 a_i}}{z_{1i}^2} \left\langle J^{a_2}(2) \ldots \widehat{J}^{a_i}(i) \ldots J^{a_n}(n) X \right\rangle$$
$$+ \frac{f^{a_1 a_i}_c}{z_{1i}} \left\langle J^{a_2}(2) \ldots J^c(i) \ldots J^{a_n}(n) X \right\rangle , \quad (7)$$

where $z_{ij} = z_i - z_j$, and $\widehat{J}$ means the current is removed. (We do not display the factors of $(-)^{[a][b]}$ in the above equation and eq. (9) below.) In order to compute F_A one only needs the integrals

$$\int \frac{d^2 \rho}{2\pi} \frac{1}{\rho - w} \frac{1}{\bar{\rho} - \bar{z}} = -\log|w - z| , \quad (8a)$$

$$\int \frac{d^2 \rho}{2\pi} \frac{1}{(\rho - w)^2} \frac{1}{\bar{\rho} - \bar{z}} = -\frac{1}{2} \frac{1}{w - z} , \quad (8b)$$

$$\int \frac{d^2 \rho}{2\pi} \frac{1}{(\rho - w)^2} \frac{1}{(\bar{\rho} - \bar{z})^2} = \frac{\pi}{2} \delta^{(2)}(w - z) , \quad (8c)$$

to systematically reduce (6) to one integral. In performing this reduction, we throw away two kinds of contributions: (i) vacuum bubbles proportional to the volume $\int d^2 x$; (ii) $\log a$ divergences that factorize into a lower order contribution to $a^{(1)}$ times a finite part which is cancelled by dividing by G_{AB}. The remaining $\log a$ divergences have a rather simple structure. Consider doing the integral $\int d1$ in (6) using (7). One is left with 3 kinds of terms $[k^2]$, $[kf]$, and $[f^2]$ terms:

$$[k^2] = \frac{\pi k^2}{2} \int d2 \ldots dn \sum_{i,j} \delta^{(2)}(z_{ij}) \eta^{a_1 a_i} \eta^{\bar{a}_1 \bar{a}_j}$$
$$\times \left\langle \ldots \widehat{J} \, \bar{J}^{\bar{a}_i}(i) \ldots J^{a_j} \, \widehat{\bar{J}}(j) \ldots X \right\rangle$$

$$[kf] = -\frac{k}{2} \int d2 \ldots dn \sum_{i,j} \left\{ \eta^{a_1 a_i} f^{\bar{a}_1 \bar{a}_j}_{\bar{c}} \frac{1}{z_{ij}} \right.$$
$$\times \left\langle \ldots \widehat{J} \, \bar{J}^{\bar{a}_i}(i) \ldots J^{a_j} \bar{J}^{\bar{c}}(j) \ldots X \right\rangle + z \to \bar{z} \right\}$$

$$[f^2] = -\int d2 \ldots dn \sum_{i,j} \log|z_{ij}| \, f^{a_1 a_i}_c f^{\bar{a}_1 \bar{a}_j}_{\bar{c}}$$
$$\times \left\langle \ldots J^c \bar{J}^{\bar{a}_i}(i) \ldots J^{a_j} \bar{J}^{\bar{c}}(j) \ldots X \right\rangle \quad (9)$$

Log a divergences arise in the following ways: (i) The $[f^2]$ term gives a log divergence when the remaining integrals give $\delta^{(2)}(z_{ij})$ by repeatedly using (8c). ($\log|z_{ij}| = \log a$ when $i = j$, (ii) Repeatedly using (8b) in the $[kf]$ terms can flip $1/z$ to $1/\bar{z}$ or vice versa a number of times. Ultimately, a Ward identity involving another f will give a $\log(z_{ij})$ like in the f^2 term. Then again, the remaining integrations must be proportional to $\delta^{(2)}(z_{ij})$ to give a log a divergence.

Integrations of $\log(z_{ij})$ with other integrands which are not δ-functions give rise to higher powers of $\log a$ but are not important for the β-function. We checked explicitly at two loops that the $\log^2 a$ divergences are squares of one-loop divergences.

The above contributions can be represented graphically, which will be useful for determining their tensor structure. Let solid lines denote the $J(z)$ OPEs and dashed lines denote the $\bar{J}(\bar{z})$ ones. Denote the f-OPEs with arrows. For the holomorphic OPEs for instance:

FIG. 1. Diagramatic representation of k- and f-OPEs.

First consider $n =$ even, which corresponds to an odd number of loops. There are two types of contributions: Type 2A, 2B. Type 2A is shown graphically in Figure 2:
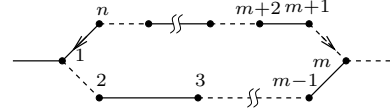

FIG. 2. Type 2A diagram.

The diagrams represent successive integrations from top to bottom. The top alternating holomorphic and anti-holomorphic lines represent succession of $1/z$ to $1/\bar{z}$ flips using (8b). The bottom alternating lines represent δ-functions coming from (8c). The rest of the diagram represents the $f^{ab}_c f^{\bar{a}\bar{b}}_{\bar{c}}$ leading to the log divergence coming from (8a). The external lines represent the left-over $J\bar{J}$ in (3). In the Type 2A diagrams, m is odd, $m = 1, 3, \ldots, n - 1$, and there are $n/2$ diagrams of this type. The Type 2B diagrams are distinguished by having the current that remains after the f^{ab}_c OPE becoming an internal part of the diagram:
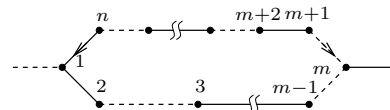

FIG. 3. Type 2B diagram.

Again m is odd, $m = 3, 5, \ldots, n - 1$. There are $(n-2)/2$ diagrams of this type.

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At $n =$ odd order, diagrams involve two holomorphic f's or two anti-holomorphic f's. There are again 2 types, 1A, 1B. The 1A diagrams are
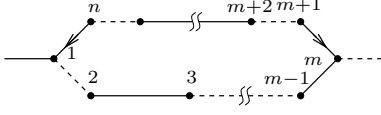

FIG. 4. Type 1A diagram.

Here $m =$ odd $= 3, 5, \ldots, n-2$. There are $(n-3)/2$ of these. A Type 1B diagram is distinguished by both f's coming to the same point:
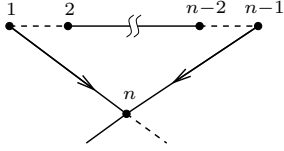

FIG. 5. Type 1B diagram.

There is one diagram of this type. It is the only contribution at two loops. For n odd each of the above diagrams has a "$z/\bar{z}$ relative" where dashed lines are replaced with solid and vice versa. These give the same contribution. The total number of diagrams is then $(n-1)$ for n even or odd.

Each diagram represents a tensor built out of η^{ab} and f_c^{ab} which must be contracted with d_{ab}^A's. An arbitrary choice of d_{ab}^A does not lead to a renormalizable theory, i.e. (3) does not hold. At one loop one needs:

$$(-)^{[b][c]} d_{ab}^A d_{cd}^B f_i^{ac} f_j^{bd} = \sum_C C_C^{AB} d_{ij}^C \quad (10)$$

for some structure constants C_C^{AB}. This closure of the operator algebra of $\mathcal{O}^A$ is equivalent to the OPE

$$\mathcal{O}^A(z, \bar{z}) \mathcal{O}^B(0) \sim \frac{1}{z\bar{z}} C_C^{AB} \mathcal{O}^C(0) \quad (11)$$

and it is well-known that C_C^{AB} determine the βeta function to one loop [7]. This one-loop condition does not ensure renormalizability at higher loops. Systematically examining the higher loop corrections we find that one also needs the quadratic form d^A to satisfy

$$\eta^{ij} d_{ai}^A d_{bj}^B = \sum_C D_C^{AB} d_{ba}^C \quad (12)$$

$$d_{ij}^A f_k^{ja} f_b^{ik} = \sum_B R_B^A \eta^{ac} d_{cb}^B \quad (13)$$

The conditions (10) and (13) guarantee that the theory is renormalizable to all orders. The conditions (13) already arise at 2 loops. One can show from the defining relations that D, C satisfy $D_C^{AB} = D_C^{BA}, C_C^{AB} = C_C^{BA}, D_D^{AC} D_E^{DB} = D_D^{AB} D_E^{DC}$.

The structure constants D, R also have a meaning in terms of OPEs. Define $T^A(z)$ as

$$T^A(z) = d_{ab}^A J^a(z) J^b(z) \quad (14)$$

suitably normal ordered. T^A is a kind of energy-momentum tensor built out of d_{ab}^A and the left-moving currents only. In the symmetric case T^A is the affine-Sugawara stress-tensor up to a normalization. Then one finds

$$T^A(z) \mathcal{O}^B(0) \sim \frac{1}{z^2} \left(2k D_C^{AB} + R_E^A D_C^{BE} \right) \mathcal{O}^C(0) , \quad (15)$$

In practice this is the most efficient way to compute the RG data. The D term is distinguished from the RD term by being proportional to k. As shown below, the βeta function depends only on the combination RD so there is no need to compute R separately. The conditions under which T^A actually define a Virasoro algebra [6] are stronger than our renormalizability conditions.

Finally we can write down the beta function. Let us arrange the couplings into a row vector $g = (g_1, g_2, \ldots)$. It is convenient to define a matrix of couplings $D(g)$, $D(g)_B^A = \sum_C D_B^{AC} g_C$. Then $gD^n(g)$ is also a row vector. The 4 kinds of contributions (coming from the Type 2A through Type 1B diagrams respectively) are

$$\beta_{g_A}^{(2A)} = -\frac{1}{2} \left(\frac{k}{2} \right)^{n-2} \left(g D^{m-1} \right)_B \left(g D^{n-m-1} \right)_C C_A^{BC}$$

$$\beta_{g_A}^{(2B)} = -\frac{1}{2} \left(\frac{k}{2} \right)^{n-2} \left(g D^{m-3} \right)_B \left(g D^{n-m-1} \right)_C C_E^{BC} (D^2)_A^E$$

$$\beta_{g_A}^{(1A)} = \left(\frac{k}{2} \right)^{n-2} \left(g D^{m-2} \right)_B \left(g D^{n-m-1} \right)_C C_E^{BC} D_A^E$$

$$\beta_{g_A}^{(1B)} = -\left(\frac{k}{2} \right)^{n-2} \left(g D^{n-2} RD \right)_A , \quad (16)$$

where $D = D(g)$. (Each diagram comes with a symmetry factor of $n!/2$.)

Remarkably, the series can be summed in a closed form. To display the result in a compact way, for any two vectors v^1 and v^2 let $C(v^1, v^2)$ denote a new row vector, $C(v^1, v^2)_A = \sum_{B,C} v_B^1 v_C^2 C_A^{BC}$. Defining $g' = g \left(1 - k^2 D^2(g)/4 \right)^{-1}$, then

$$\beta_g = -\frac{1}{2} C(g', g') \left(1 + k^2 D^2/4 \right)$$
$$+ \frac{k^3}{8} C(g'D, g'D) D - \frac{k}{2} g' DRD , \quad (17)$$

where again $D = D(g)$. The above βeta function determines how couplings flow with increasing *length* scale a, where large a corresponds to the low-energy limit.

The simplest possible case corresponds to a single coupling with $d_{ab} = \eta_{ab}$. Normalizing the Lie algebra generators in the fundamental representation F so that $\text{Tr}(t^a t^b) = c_F \delta^{ab} = \eta^{ab}$ and using $\eta_{ij} f_k^{jc} f_d^{ik} = C_{\text{adj}} \delta_d^c$, one finds

$$\beta_g = \frac{1}{2} \frac{C_{\text{adj}} g^2}{(1 + k g c_F/2)^2} . \quad (18)$$

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This agrees with previous two-loop calculations for the Gross-Neveu model [8]. (The usual convention corresponds to $c_F = 1/2$ with $k = 1$.) A three-loop computation was performed in [9].

From the result (18) we see that the βeta function is identically zero if $C_{\text{adj}} = 0$. This occurs for the superalgebras $osp(2n + 2|2n)$ and $PSL(n|n)$ [10–12]. In the work [12], an all orders βeta function was proposed in the case of $gl(1|1)$ using a simple scaling argument, and the result had a structure of the kind described here.

The renormalizability conditions (11,15) are rather restrictive. Consider for instance a completely anisotropic perturbation $\sum_a g_a J^a \bar{J}^a = \sum_a g_a \mathcal{O}^a$ in a basis where $\eta^{ab} = \delta^{ab}$, i.e. the couplings g_a, g_b are not equal for $a \neq b$. This is not renormalizable in general since the operator algebra of $\mathcal{O}^a$ does not close:

$$\mathcal{O}^a(z, \bar{z}) \mathcal{O}^b(0) \sim \frac{1}{z\bar{z}} \sum_{c,d} f_c^{ab} f_d^{ab} J^c \bar{J}^d(0) \qquad (19)$$

The algebra only closes if $[t^a, t^b]$ is proportional to a *single* generator, since then $f_c^{ab} f_d^{ab} \propto \delta_{cd}$ and one has $C_c^{ab} = (f_c^{ab})^2$. Adopting a basis for J^a based on the root system of the Lie algebra, in order to build a renormalizable theory one needs to include additional interactions of the form $(\alpha \cdot H)(\alpha' \cdot \bar{H})$ where α, α' are roots and H is in the Cartan basis. The resulting theory is renormalizable and one can then compute the RG data C, D, R and the βeta function. Interesting models which are still renormalizable can be obtained by equating subsets of the couplings consistent with a global sub-symmetry.

The simplest example is $su(2)$ since it only has one Cartan generator. Let us illustrate our main result in this example. Let the currents be normalized as

$$J_3(z) J^\pm(0) \sim \pm \frac{1}{z} J^\pm(0), \ \ J^+(z) J^-(0) \sim \frac{k}{2} \frac{1}{z^2} + \frac{1}{z} J_3(0),$$
$$J_3(z) J_3(0) \sim \frac{k}{2} \frac{1}{z^2} \ . \qquad (20)$$

This corresponds to $\eta^{ab} = \frac{1}{2}\delta^{ab}$, $C_{\text{adj}} = 2$, and $c_F = 1/2$. Consider the interaction

$$\sum_A g_A \mathcal{O}^A = g_\perp \left(J^+ \bar{J}^- + J^- \bar{J}^+ \right) + g_\parallel J_3 \bar{J}_3 \ . \qquad (21)$$

Using (11) and (15) the RG data are: $D_\perp^{\perp\perp} = D_\parallel^{\parallel\parallel} = 1/2$, $C_\perp^{\perp\parallel} = C_\perp^{\parallel\perp} = -1$, $C_\parallel^{\perp\perp} = -2$, $(RD)_\perp^{\perp\perp} = (RD)_\perp^{\parallel\perp} = 1$, $(RD)_\parallel^{\perp\parallel} = 2$. After some algebra one finds

$$\beta_{g_\perp} = \frac{g_\perp \left(g_\parallel - k g_\perp^2/4 \right)}{\left(1 - k^2 g_\perp^2/16\right) \left(1 + k g_\parallel/4\right)}$$
$$\beta_{g_\parallel} = \frac{g_\perp^2 \left(1 - k g_\parallel/4\right)^2}{\left(1 - k^2 g_\perp^2/16\right)^2} \ . \qquad (22)$$

When $g_\perp = g_\parallel$ one recovers (18).

One can use these βeta functions to study Kosterlitz-Thouless flows [13] at strong coupling, or the issue of symmetry restoration recently addressed in [14]. One observes flows to the isotropic line $g_\perp = g_\parallel$ in the regime $0 < g_{\perp,\parallel} < 4/k$. However there are other features which will be reported elsewhere, along with applications of the above results to disordered fermions [15].

We hope to publish a longer version of this paper with more examples [16].

ACKNOWLEDGMENTS

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